

LARGE 2-ADIC GALOIS IMAGE AND NON-EXISTENCE OF CERTAIN ABELIAN SURFACES OVER \mathbb{Q}

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ABSTRACT. Motivated by our arithmetic applications, we required some tools that might be of independent interest.

Let \mathcal{E} be an absolutely irreducible group scheme of rank p^4 over \mathbb{Z}_p . We provide a complete description of the Honda systems of p -divisible groups \mathcal{G} such that $\mathcal{G}[p^{n+1}]/\mathcal{G}[p^n] \simeq \mathcal{E}$ for all n . Then we find a bound for the abelian conductor of the second layer $\mathbb{Q}_p(\mathcal{G}[p^2])/\mathbb{Q}_p(\mathcal{G}[p])$, stronger in our case than can be deduced from Fontaine's bound.

Let $\pi: \mathrm{Sp}_{2g}(\mathbb{Z}_p) \rightarrow \mathrm{Sp}_{2g}(\mathbb{F}_p)$ be the reduction map and let G be a closed subgroup of $\mathrm{Sp}_{2g}(\mathbb{Z}_p)$ with $\overline{G} = \pi(G)$ irreducible and generated by transvections. We fill a gap in the literature by showing that if $p = 2$ and G contains a transvection, then G is as large as possible in $\mathrm{Sp}_{2g}(\mathbb{Z}_p)$ with given reduction \overline{G} , i.e. $G = \pi^{-1}(\overline{G})$.

One simple application arises when $A = J(C)$ is the Jacobian of a hyperelliptic curve $C: y^2 + Q(x)y = P(x)$, where $Q(x)^2 + 4P(x)$ is irreducible in $\mathbb{Z}[x]$ of degree $m = 2g + 1$ or $2g + 2$, with Galois group $S_m \subset \mathrm{Sp}_{2g}(\mathbb{F}_2)$. If the Igusa discriminant I_{10} of C is odd and some prime q exactly divides I_{10} , then $G = \mathrm{Gal}(\mathbb{Q}(A[2^\infty])/\mathbb{Q})$ is $\tilde{\pi}^{-1}(S_m)$, where $\tilde{\pi}: \mathrm{GSp}_{2g}(\mathbb{Z}_p) \rightarrow \mathrm{Sp}_{2g}(\mathbb{F}_p)$.

When $m = 5$, $Q(x) = 1$ and $I_{10} = N$ is a prime, $A = J(C)$ is an example of a *favorable* abelian surface. We use the machinery above to obtain non-existence results for certain favorable abelian surfaces, even for large N .

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1. INTRODUCTION

Let A/\mathbb{Q} be a g -dimensional abelian variety and $G = \text{Gal}(\mathbb{Q}(A[p^\infty])/\mathbb{Q})$ the Galois group of its p -division tower. Serre's work on the open image problem for abelian varieties has stimulated a large literature. For instance, [ALS, H, KM, V, ZS] show that, under suitable hypotheses, G is an open subgroup of $\text{GSp}_{2g}(\mathbb{Z}_p)$, at least for large p . We concentrate on $p = 2$, for which more residual images exist. Theorem 2.1.1 describes our group theoretical conclusions. As usual, suitable abelian varieties thereby give rise to large Galois extensions with controlled ramification, as in Proposition 2.3.4.

Given an integer N and a group scheme \mathcal{E} over $\mathbb{Z}[\frac{1}{N}]$ of exponent p , one may ask for the existence (or even the uniqueness up to isogeny) of an abelian variety A with $A[p] \simeq \mathcal{E}$. In [BK1], we found non-existence criteria when \mathcal{E} is reducible. In this paper, we treat non-existence criteria when $\dim A = 2$ and $\mathcal{E}_{|\mathbb{Z}_p}$ is absolutely irreducible. Then A has a polarization of degree prime to p and $\text{Gal}(\mathbb{Q}(A[p^\infty])/\mathbb{Q})$ is contained in $\text{GSp}_4(\mathbb{Z}_p)$. This requires a delicate study of the extensions \mathcal{W} of \mathcal{E} by \mathcal{E} of exponent p^2 . Non-existence of $\mathbb{Q}(\mathcal{W})/\mathbb{Q}$ implies that of A with $A[p^2] \simeq \mathcal{W}$.

Let \mathcal{E} be an absolutely irreducible group scheme of rank p^4 over \mathbb{Z}_p . In §4, we give a complete description of the Honda systems of p -divisible groups \mathcal{G} such that $\mathcal{G}[p^{n+1}]/\mathcal{G}[p^n] \simeq \mathcal{E}$ for all n . In §5, we study the field of points of $\mathcal{W} = \mathcal{G}[p^2]$, thereby obtaining a bound for the abelian conductor of $\mathbb{Q}_p(\mathcal{W})/\mathbb{Q}_p(\mathcal{E})$, stronger in our special case than can be deduced from Fontaine's bound. When A is the Jacobian of a genus 2 curve over \mathbb{Q}_2 , the parameters associated to $A[4]$ are determined in Proposition 5.3.1.

For the global applications in §6, let $p = 2$ and recall the following definition.

Definition 1.1 ([BK3]). A quintic field is *favorable* if its discriminant is $\pm 16N$ with N prime and the ramification index over the prime 2 is 5. An abelian surface A/\mathbb{Q} of conductor N is *favorable* if its 2-division field is the Galois closure of a favorable quintic field.

If A is favorable, then the image in $\text{Sp}_4(\mathbb{F}_2)$ of the representation of $G_{\mathbb{Q}}$ on $A[2]$ is $O_4^-(\mathbb{F}_2) \simeq \mathcal{S}_5$, with transvections corresponding to transpositions. In addition, $A[2]$ is biconnected, absolutely simple and Cartier self-dual over \mathbb{Z}_2 . Let $\mathfrak{S} = \pi^{-1}(\mathcal{S}_5)$, where $\pi : \text{GSp}_4(\mathbb{Z}/4\mathbb{Z}) \rightarrow \text{Sp}_4(\mathbb{F}_2)$ is the natural projection. By Remark 6.1, $\mathbb{Q}(A[4])$ is a *favorable* \mathfrak{S} -field with $F = \mathbb{Q}(A[2])$, as in the following definition.

Definition 1.2. Fix the Galois closure F of a favorable quintic field of discriminant $\pm 16N$ and let L be a field containing F . Then L is a *favorable* \mathfrak{S} -field if L/\mathbb{Q} is Galois, with $\text{Gal}(L/\mathbb{Q}) \simeq \mathfrak{S}$ and

- i) L/\mathbb{Q} is unramified outside $2N\infty$;
- ii) the abelian conductor exponent at primes over 2 in L/F is 6 and
- iii) the inertia group at each prime over N in L/\mathbb{Q} is generated by a transvection.

Proposition 6.2 gives a testable ray class field criterion necessary for the existence of a favorable \mathfrak{S} -field L . This explains the non-existence results in [BK3, Table 3].

2. A LARGE IMAGE RESULT

2.1. Review. For closed subgroups Γ of $\text{GL}_m(\mathbb{Z}_p)$, set $\Gamma^{(n)} = \{g \in \Gamma \mid g \equiv I_m(p^n)\}$ and $\bar{\Gamma} = \Gamma/\Gamma^{(1)}$. A closed subgroup G of Γ is *saturated in* Γ if $G^{(1)} = \Gamma^{(1)}$, so that

G is as large as possible in Γ , subject to its reduction being \overline{G} . When there is a symplectic pairing $[\cdot, \cdot]: M \times M \rightarrow \mathbb{Z}_p$ on $M = \mathbb{Z}_p^{2g}$, *transvections* in $\mathrm{Sp}(M)$ have the form $\sigma(x) = x - \lambda[y, x]y$, with y in M and λ in \mathbb{Z}_p^\times .

Theorem 2.1.1. *Let G be a closed subgroup of $\mathrm{Sp}_{2g}(\mathbb{Z}_p)$ containing transvections. If \overline{G} is irreducible and generated by transvections, then G is saturated in $\mathrm{Sp}_{2g}(\mathbb{Z}_p)$.*

This assertion is well-known when $\overline{G} = \mathrm{Sp}_{2g}(\mathbb{F}_p)$, so we need only consider $p = 2$ and proper subgroups, thanks to a classical result of McLaughlin.

Proposition 2.1.2 ([McL]). *For $g \geq 2$, let H be an irreducible proper subgroup of $\mathrm{Sp}_{2g}(\mathbb{F}_p)$ generated by transvections. Then $p = 2$ and H is one of the following:*

- i) *the symmetric group \mathcal{S}_m with $m = 2g + 1$ or $2g + 2$,*
- ii) *the orthogonal group $O_{2g}^+(\mathbb{F}_2)$ with $g \geq 3$, or $O_{2g}^-(\mathbb{F}_2)$.*

Orthogonal groups and theta characteristics are reviewed in [D, GH, BK2]. Set $\mathfrak{sp}_{2g}(\mathbb{F}_p) = \{A \in \mathrm{Mat}_{2g}(\mathbb{F}_p) \mid A^t J + J A = 0\}$, where J is the *Gram matrix* of a basis e_1, \dots, e_{2g} for M . Then $\dim_{\mathbb{F}_p} \mathfrak{sp}_{2g}(\mathbb{F}_p) = 2g^2 + g$, since

$$(2.1.3) \quad \mathfrak{sp}_{2g}(\mathbb{F}_p) = \left\{ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid b = b^t, c = c^t, d = -a^t \right\} \quad \text{when } J = \begin{bmatrix} 0_g & I_g \\ -I_g & 0_g \end{bmatrix}.$$

To prove the Theorem, one verifies that sending an element $1 + pA$ of $G^{(1)}$ to $A \bmod p$ induces an isomorphism $\mathcal{L}: G^{(1)}/G^{(2)} \rightarrow \mathfrak{sp}_{2g}(\mathbb{F}_p)$. It follows that $1 + p^n A \mapsto A \bmod p$ gives an isomorphism $G^{(n)}/G^{(n+1)} \rightarrow \mathfrak{sp}_{2g}(\mathbb{F}_p)$ for all $n \geq 1$. Then one shows by induction that $G^{(1)} \rightarrow \mathrm{Sp}_{2g}^{(1)}(\mathbb{Z}/p^n\mathbb{Z})$ is surjective and passes to the limit. For the groups in the Theorem, the transvections s in \overline{G} form one conjugacy class. Since G contains a transvection, s lifts to a transvection σ in G , say $\sigma(x) = x - \lambda[y, x]y$. Furthermore, $\mathcal{L}(\sigma^p) \in \mathfrak{sp}_{2g}(\mathbb{F}_p)$ is the matrix representation of the endomorphism of $\overline{M} = M/pM$ given by

$$(2.1.4) \quad f_{\overline{y}}: \overline{x} \mapsto (s - 1)(\overline{x}) = -\lambda[\overline{y}, \overline{x}]\overline{y},$$

where $x \mapsto \overline{x}$ denotes the projection map $M \rightarrow \overline{M}$ and the pairing is induced on \overline{M} . Hence it suffices to show that the $f_{\overline{y}}$'s generate $\mathfrak{sp}_{2g}(\mathbb{F}_p)$. This is done in Lemmas 2.2.1 and 2.3.3.

2.2. The case $\overline{G} = \mathcal{S}_m$. Let \mathcal{S}_m act by permuting the coordinates of

$$W = \{(a_1, \dots, a_m) \in \mathbb{F}_2^m \mid a_1 + \dots + a_m = 0\}$$

and let $V = W/\{(a, \dots, a) \mid a \in \mathbb{F}_2\}$ or $V = W$ if m is even or odd, respectively. Then \overline{G} is symplectic for the pairing on V induced by $[(a_i), (b_i)] = \sum_1^m a_i b_i$.

Let v_{ij} in V be represented by the vector in W with non-zero entries only in coordinates i and j . For x in V , the transvection $s_{ij}(x) = x + [v_{ij}, x]v_{ij}$ corresponds to $\pi = (ij)$ in \mathcal{S}_m . Let $f_{ij}(x) = (s_{ij} - 1)(x) = [v_{ij}, x]v_{ij}$ be the endomorphism of V as in (2.1.4).

Lemma 2.2.1. *The set $\{f_{ij} \mid 1 \leq i < j \leq 2g + 1\}$ spans $\mathfrak{sp}_{2g}(\mathbb{F}_2)$.*

Proof. The vectors $b_i = v_{i, 2g+1}$ for $1 \leq i \leq 2g$ form a basis for V . By definition, $v_{ij} = b_i + b_j$ for $1 \leq i < j \leq 2g$. To prove that the $2g^2 + g$ elements in the Lemma

span, we show that they are linearly independent. If not, there are constants $\alpha_{ij}, \beta_i \in \mathbb{F}_2$ satisfying

$$\sum_{1 \leq i < j \leq 2g} \alpha_{ij} [x, v_{ij}] v_{ij} + \sum_{i=1}^{2g} \beta_i [x, b_i] b_i = 0 \quad \text{for all } x \in V.$$

Fix k in $\{1, \dots, 2g\}$ and evaluate at $x = b_k$, using

$$[b_k, v_{ij}] = \begin{cases} 1 & \text{if } i = k \text{ or } j = k, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad [b_k, b_i] = \begin{cases} 1 & \text{if } i \neq k, \\ 0 & \text{if } i = k, \end{cases}$$

to obtain

$$(2.2.2) \quad \sum_{j > k} \alpha_{kj} (b_k + b_j) + \sum_{i < k} \alpha_{ik} (b_i + b_k) + \sum_{i \neq k} \beta_i b_i = 0.$$

Match coefficients of b_i with $i < k$ to find that $\alpha_{ik} = \beta_i$ for all $i < k$ and those of b_j with $j > k$ to find that $\alpha_{kj} = \beta_j$ for all $j > k$. Hence $\alpha_{ij} = \beta_i = \beta_j = \gamma$ is constant. From the coefficients of b_k in (2.2.2) we have $(2g-1)\gamma b_k = 0$ and so $\gamma = 0$. \square

2.3. The case $\overline{G} \simeq O_{2g}^\pm(\mathbb{F}_2)$. Let V be a symplectic space of dimension $2g$ over \mathbb{F}_2 with basis $\{b_i\}$ and Gram matrix J in (2.1.3). Consider the theta characteristic

$$(2.3.1) \quad \theta_\epsilon(x) = Q_\epsilon(x_g, x_{2g}) + \sum_{i=1}^{g-1} x_i x_{g+i} \quad \text{for } x = (x_1, \dots, x_{2g}) \text{ in } V,$$

with $Q_+(x_g, x_{2g}) = x_g x_{2g}$ in the even case and $Q_-(x_g, x_{2g}) = x_g^2 + x_g x_{2g} + x_{2g}^2$ in the odd case. We have $\theta_\epsilon(x+y) = \theta_\epsilon(x) + \theta_\epsilon(y) + [x, y]$, i.e. θ_ϵ belongs to the pairing $[\cdot, \cdot]$ on V . The transvection $s: x \mapsto x + [v, x]v$ in $\text{Sp}_{2g}(\mathbb{F}_2)$ acts on theta characteristics by $s(\theta)(x) = \theta(x) + (1 + \theta(v))[v, x]^2$. Hence s is in the stabilizer $O_{2g}^\epsilon(\mathbb{F}_2)$ of θ_ϵ in $\text{Sp}_{2g}(\mathbb{F}_2)$ exactly when $\theta_\epsilon(v) = 1$.

Let $f_v: x \mapsto [v, x]v$ be the endomorphism of V as in (2.1.4). For $\epsilon = \pm$, let

$$\mathcal{F}_g^\epsilon(V) = \{f_v \in \mathfrak{sp}_{2g}(\mathbb{F}_2) \mid v \in V \text{ and } \theta_\epsilon(v) = 1\}$$

and write $\langle \mathcal{F}_g^\epsilon(V) \rangle$ for its span in $\mathfrak{sp}_{2g}(\mathbb{F}_2)$.

Remark 2.3.2. $\langle \mathcal{F}_1^-(V) \rangle = \mathfrak{sp}_2(\mathbb{F}_2) = \{A \in \text{Mat}_2(\mathbb{F}_2) \mid \text{trace}(A) = 0\}$, with

$$f_{b_1} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad f_{b_2} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad f_{b_1+b_2} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Also, $\dim \langle \mathcal{F}_1^+(V) \rangle = 1$, $\dim \langle \mathcal{F}_2^+(V) \rangle = 6$ and $\langle \mathcal{F}_3^+(V) \rangle = \mathfrak{sp}_6(\mathbb{F}_2)$ is 21-dimensional. There are symplectic isomorphisms $O_6^+(\mathbb{F}_2) \simeq \mathcal{S}_8$ and $O_4^-(\mathbb{F}_2) \simeq \mathcal{S}_5$.

Lemma 2.3.3. *We have $\langle \mathcal{F}_g^\epsilon(V) \rangle = \mathfrak{sp}_{2g}(\mathbb{F}_2)$ for $\begin{cases} g \geq 1 & \text{if } \epsilon = -, \\ g \geq 3 & \text{if } \epsilon = +. \end{cases}$*

Proof. For $g \geq 2$, the subspace $V_1 = \text{span}\{b_j \mid j \neq 1, g+1\}$ of V is isomorphic to the symplectic space of dimension $2g-2$ whose theta characteristic θ'_ϵ and pairing are obtained by restriction from θ_ϵ and $[\cdot, \cdot]$. Given a linear map $h_1: V_1 \rightarrow V_1$, let $h = \delta_1(h_1)$ denote its extension to V satisfying $h(b_1) = h(b_{g+1}) = 0$. Then $\delta_1: \mathfrak{sp}_{2g-2}(\mathbb{F}_2) \hookrightarrow \mathfrak{sp}_{2g}(\mathbb{F}_2)$. In particular, $\delta_1(x' \mapsto [v', x']v')$ is the matrix of the linear map $f_{v'}$ on V given by $f_{v'}(x) = [v', x]v'$. Hence $Y_1 = \delta_1(\mathcal{F}_g^\epsilon(V_1))$ is contained in $\mathcal{F}_g^\epsilon(V)$.

Remark 2.3.2 treats the small values of g , so we assume $g \geq 2$ for $\mathcal{F}_g^-(V)$ and $g \geq 4$ for $\mathcal{F}_g^+(V)$. Let $V_2 = \text{span}\{e_j \mid j \neq 2, g+2\}$ and extend $h_2 : V_2 \rightarrow V_2$ to a linear map $h = \delta_2(h_2)$ on V by setting $h(b_2) = h(b_{g+2}) = 0$. Then $\delta_2 : \mathfrak{sp}_{2g-2}(\mathbb{F}_2) \hookrightarrow \mathfrak{sp}_{2g}(\mathbb{F}_2)$ and $Y_2 = \delta_2(\mathcal{F}_g^\epsilon(V_2))$ also is contained in $\mathcal{F}_g^\epsilon(V)$. By the induction hypothesis,

$$\dim Y_i = \dim \mathfrak{sp}_{2g-2}(\mathbb{F}_2) = 2(g-1)^2 + (g-1) = 2g^2 - 3g + 1 \quad \text{for } i = 1, 2.$$

We have $\dim Y_1 \cap Y_2 \leq \dim \mathfrak{sp}_{2g-4}(\mathbb{F}_2) = 2(g-2)^2 + g - 2 = 2g^2 - 7g + 6$. But $\dim \mathfrak{sp}_{2g}(\mathbb{F}_2) = 2g^2 + g$, so the codimension of $Y_1 + Y_2$ in $\mathfrak{sp}_{2g}(\mathbb{F}_2)$ is at most 4.

Next we fill in a 4-dimensional subspace of $\mathfrak{sp}_{2g}(\mathbb{F}_2)$ independent of $Y_1 + Y_2$. For matrices in Y_k , all entries in the row or column numbered k or $g+k$ are 0. If A is in $Y_1 + Y_2$, then $A_{ij} = 0$ for the eight pairs (i, j) with

$$\{i, j\} = \{1, 2\}, \{1, g+2\}, \{g+1, 2\} \text{ or } \{g+1, g+2\}.$$

By the symplectic condition (2.1.3) on A , we have

$$A_{1,2} = A_{g+2,g+1}, \quad A_{1,g+2} = A_{2,g+1}, \quad A_{g+1,2} = A_{g+2,1}, \quad A_{g+1,g+2} = A_{2,1}.$$

Define $j^* = j + g$ if $j < g$ and $j^* = j - g$ otherwise. Fix (i, j) in

$$S = \{(1, 2), (1, g+2), (g+1, 2), (g+1, g+2)\}$$

and let $v = v_{ij} = b_i + b_{j^*} + b_g + b_{2g}$. Since $\theta_\epsilon(v) = 1$, the linear map f_v is in $\mathcal{F}_g^\epsilon(V)$. Also, $A_{i,j} = 1$ and $A_{i',j'} = 0$ for all other pairs (i', j') in S . Hence the f_v 's generate the desired 4-dimensional space as (i, j) ranges over the 4 pairs in S . \square

In the following Proposition, a group is said to be McL if it is isomorphic to $\text{Sp}_{2g}(\mathbb{F}_2)$ or one of the groups in Proposition 2.1.2.

Proposition 2.3.4. *Let A/\mathbb{Q} be a g -dimensional abelian variety of odd conductor N , with $q \parallel N$ for a prime q ramifying in $F = \mathbb{Q}(A[2])$ and let $F_\infty = \mathbb{Q}(A[2^\infty])$. If $\overline{G} = \text{Gal}(F/\mathbb{Q})$ is McL, then $G = \text{Gal}(F_\infty/\mathbb{Q})$ is saturated in $\text{GSp}_{2g}(\mathbb{Z}_2)$. Also, \overline{G} is McL if $A[2]$ is irreducible and one of the following holds:*

- i) *the conductor of $A[2]$ is square-free and $\sqrt{-1}$ is not in F , or*
- ii) *F contains no proper extension of \mathbb{Q} unramified at q .*

Proof. Since $A[2]$ is irreducible in all cases, a minimal polarization of A has odd degree. Hence the Weil pairing induces a perfect pairing on the Tate module $T_2(A)$ and G is a closed subgroup of $\text{GSp}_{2g}(\mathbb{Z}_2)$. The symplectic similitude $\nu : G \rightarrow \mathbb{Z}_2^\times$ giving the action of G on μ_{2^∞} is surjective and so F_∞ contains $\mathbb{Q}(\mu_{2^\infty})$. By Grothendieck's monodromy theorem, inertia at $v|q$ is generated topologically by a transvection σ_v , since the toroidal dimension of A at q is 1 and q ramifies in F .

Assume (i). For each prime w dividing $\text{cond}(A[2])$, inertia at w is generated by a transvection s_w in \overline{G} . Thus the fixed field k of the normal subgroup generated by all s_w is unramified outside 2∞ . As in [BK2, Prop. 6.2], Fontaine's bound on ramification at 2 implies that k is contained in $\mathbb{Q}(i)$ and thus $k = \mathbb{Q}$. Hence \overline{G} is generated by transvections and so \overline{G} is McL. In case (ii), the subfield of F fixed by the normal closure of $\overline{\sigma}_v$ in \overline{G} is unramified over q , so equals \mathbb{Q} and \overline{G} is McL.

If \overline{G} is McL, transvections form one conjugacy class generating \overline{G} . Since σ_v fixes $F \cap \mathbb{Q}(\mu_{2^\infty})$, the latter equals \mathbb{Q} and the restriction of ν to $G^{(1)} = \text{Gal}(F_\infty/F)$ surjects onto \mathbb{Z}_2^\times . For $H = G \cap \text{Sp}_{2g}(\mathbb{Z}_2)$, we find that $\overline{H} = \overline{G}$ so $H^{(1)} = \text{Sp}_{2g}^{(1)}(\mathbb{Z}_2)$ by Theorem 2.1.1. Hence $G^{(1)} = \text{GSp}_{2g}^{(1)}(\mathbb{Z}_2)$. \square

Example 2.3.5. Let A be the Jacobian of a hyperelliptic curve $y^2 + Q(x)y = P(x)$, where $Q(x)^2 + 4P(x)$ is irreducible in $\mathbb{Z}[x]$ of degree $m = 2g + 1$ or $2g + 2$, with Galois group $\mathcal{S}_m \subset \mathrm{Sp}_{2g}(\mathbb{F}_2)$. If the Igusa discriminant I_{10} of C is odd and some prime q exactly divides I_{10} , then $G = \mathrm{Gal}(\mathbb{Q}(A[2^\infty])/\mathbb{Q})$ is saturated in $\mathrm{GSp}_{2g}(\mathbb{Z}_2)$.

3. PRELIMINARIES ON HONDA SYSTEMS

The basic material on Honda systems may be found in [BC, C2, F1] and is summarized in [BK3]. We review the required notation and recall some finite Honda systems constructed in [BK3].

Let k a perfect field of prime characteristic p and $\mathbb{W} = \mathbb{W}(k)$ the ring of Witt vectors over k . Fix an algebraic closure \overline{K} of the field of fractions K of \mathbb{W} , let $\overline{\mathbb{W}}$ be its ring of integers and write $G_K = \mathrm{Gal}(\overline{K}/K)$. Let $\sigma: \mathbb{W} \rightarrow \mathbb{W}$ be the Frobenius automorphism characterized by $\sigma(x) \equiv x^p \pmod{p}$ for x in \mathbb{W} . The Dieudonné ring $D_k = \mathbb{W}[F, V]$ is generated by the Frobenius operator F and Verschiebung operator V , with $FV = VF = p$, $Fa = \sigma(a)F$ and $Va = \sigma^{-1}(a)V$ for all a in \mathbb{W} .

A *Honda system* consists of a finitely generated free \mathbb{W} -module \mathcal{M} , a submodule \mathcal{L} of \mathcal{M} and a Frobenius-semilinear injective endomorphism F of \mathcal{M} such that $p\mathcal{M} \subseteq F\mathcal{M}$ and inclusion induces an isomorphism $\mathcal{L}/p\mathcal{L} \rightarrow \mathcal{M}/F\mathcal{M}$. Then \mathcal{M} becomes a D_k -module with *Verschiebung* defined by $Vx = F^{-1}(px)$ for all x in \mathcal{M} .

Let \mathcal{M} be a D_k -module, finitely generated and free as a \mathbb{W} -module and let \mathcal{L} be a \mathbb{W} -submodule of \mathcal{M} . Then $(\mathcal{M}, \mathcal{L})$ is a Honda system if and only if the following sequence is exact:

$$(3.1) \quad 0 \rightarrow \mathcal{L} \xrightarrow{V} \mathcal{M} \xrightarrow{F} \mathcal{M}/\mathcal{L} \rightarrow 0.$$

Lemma 3.2. *Given a Honda system $(\mathcal{M}, \mathcal{L})$, let $\mathcal{M}^* = \mathrm{Hom}_{\mathbb{W}}(\mathcal{M}, \mathbb{W})$ and let \mathcal{L}^* be the annihilator of \mathcal{L} in \mathcal{M}^* . Define F and V on elements ψ of \mathcal{M}^* by*

$$(3.3) \quad F(\psi)(x) = \sigma(\psi(Vx)) \quad \text{and} \quad V(\psi)(x) = \sigma^{-1}(\psi(Fx)) \quad \text{for all } x \in \mathcal{M}.$$

Then $(\mathcal{M}^, \mathcal{L}^*)$ forms a Honda system.*

Proof. Since $\mathcal{L} \cap F\mathcal{M} = p\mathcal{L}$, the quotient \mathcal{M}/\mathcal{L} is torsion-free, so \mathcal{L} is a direct summand of \mathcal{M} . The pairing $\langle -, - \rangle: \mathcal{M}^* \times \mathcal{M} \rightarrow \mathbb{W}$ induces perfect pairings:

$$\mathcal{M}^*/\mathcal{L}^* \times \mathcal{L} \rightarrow \mathbb{W} \quad \text{and} \quad \mathcal{L}^* \times \mathcal{M}/\mathcal{L} \rightarrow \mathbb{W}.$$

By dualizing (3.1), the sequence $0 \rightarrow \mathcal{L}^* \xrightarrow{V} \mathcal{M}^* \xrightarrow{F} \mathcal{M}^*/\mathcal{L}^* \rightarrow 0$ is exact. \square

If F is topologically nilpotent, then $(\mathcal{M}, \mathcal{L})$ is *connected*. If both F and V are topologically nilpotent, then $(\mathcal{M}, \mathcal{L})$ is *biconnected*.

A *finite Honda system* is a pair (M, L) consisting of a D_k -module M of finite \mathbb{W} -length and a \mathbb{W} -submodule L with $V: L \rightarrow M$ injective and the map $L/pL \rightarrow M/FM$ induced by the identity is an isomorphism. If $(\mathcal{M}, \mathcal{L})$ is a Honda system then $(\mathcal{M}/p^n\mathcal{M}, \mathcal{L}/p^n\mathcal{L})$ is a finite Honda system.

Let \widehat{CW}_k denote the formal k -group scheme associated to the *Witt covector* group functor CW_k , cf. [C2, F2]. In particular, if k' is a finite extension of k and K' is the field of fractions of $W(k')$, we have $CW_k(k') \simeq K'/W(k')$. If R is a k -algebra, then $D_k = \mathbb{W}[F, V]$ acts on elements $\mathbf{a} = (\dots, a_{-n}, \dots, a_{-1}, a_0)$ of $CW_k(R)$ by $F\mathbf{a} = (\dots, a_{-n}^p, \dots, a_{-1}^p, a_0^p)$, $V\mathbf{a} = (\dots, a_{-(n+1)}, \dots, a_{-2}, a_{-1})$ and $\dot{c}\mathbf{a} = (\dots, c^{p^{-n}}a_{-n}, \dots, c^{p^{-1}}a_{-1}, ca_0)$, where \dot{c} is the Teichmüller lift of c . It is

convenient to write $(\vec{0}, a_{-n}, \dots, a_0)$ for an element of $\widehat{CW}_k(\overline{\mathbb{W}}/p\overline{\mathbb{W}})$ with $a_{-m} = 0$ for all $m > n$.

The Hasse-Witt exponential map is a homomorphism of additive groups:

$$\xi : \widehat{CW}_k(\overline{\mathbb{W}}/p\overline{\mathbb{W}}) \rightarrow \overline{K}/p\overline{\mathbb{W}} \quad \text{given by} \quad (\dots, a_{-n}, \dots, a_{-1}, a_0) \mapsto \sum p^{-n} \tilde{a}_{-n}^{p^n}$$

independent of the choice of lifts \tilde{a}_{-n} in $\overline{\mathbb{W}}$.

We generally use calligraphic letters, e.g. \mathcal{V} for finite flat group schemes and the corresponding roman letter, e.g. V for the associated Galois module. If (M, L) is the finite Honda system of \mathcal{V} , the points of V correspond to D_k -homomorphisms $\varphi : M \rightarrow \widehat{CW}_k(\overline{\mathbb{W}}/p\overline{\mathbb{W}})$ such that $\xi(\varphi(L)) = 0$, with the action of G_K on V induced from its action on $\widehat{CW}_k(\overline{\mathbb{W}}/p\overline{\mathbb{W}})$.

For the study of p -divisible groups in the next section, recall the finite Honda system \mathfrak{E}_λ introduced in [BK3, §4] and our classification of extensions of exponent p of \mathfrak{E}_λ by \mathfrak{E}_λ .

Notation 3.4. Fix λ in k^\times and let $\mathfrak{E}_\lambda = (M, L)$ be the finite Honda system with a standard k -basis x_1, x_2, x_3, x_4 for M such that $L = \text{span}\{x_1, x_2\}$ and Verschiebung and Frobenius are represented by the matrices:

$$V = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Denote the corresponding group scheme by \mathcal{E}_λ and its Galois module by E_λ .

Proposition 3.5 ([BK3, Prop. 5.1.1]). *Let $\dot{\lambda}$ be the Teichmüller lift of λ and let $\mathfrak{R}_\lambda = \{a \in \overline{\mathbb{W}}/p\overline{\mathbb{W}} \mid \lambda^{p^2} a^{p^4} \equiv (-p)^{p+1} a \pmod{p^{p+2}\overline{\mathbb{W}}}\}$. For a in \mathfrak{R}_λ , define $b = b_a$ and $c = c_a$ in $\overline{\mathbb{W}}/p\overline{\mathbb{W}}$ by $b \equiv -\frac{1}{p}\lambda^p a^{p^3} \pmod{p\overline{\mathbb{W}}}$ and $c \equiv \lambda a^{p^2} \pmod{p\overline{\mathbb{W}}}$.*

- i) *Let x_1, \dots, x_4 be a standard basis for the finite Honda system \mathfrak{E}_λ of \mathcal{E}_λ . A D_k -map ψ represents a point of \mathcal{E}_λ if and only if $\psi(x_1) = (\vec{0}, c, b, a)$ for some a in \mathfrak{R}_λ . If so, $\psi(x_2) = (\vec{0}, c, b)$, $\psi(x_3) = (\vec{0}, \lambda^{-1}c)$ and $\psi(x_4) = (\vec{0}, a^p)$.*
- ii) *$F = K(E_\lambda)$ is the splitting field of $\dot{\lambda}^{p^2} x^{p^4-1} - (-p)^{p+1}$ over K . The maximal subfield of F unramified over K is $F_0 = K(\mu_{p^4-1}, \xi)$, where ξ is any root of $x^{p+1} - \dot{\lambda}$. Moreover F/F_0 is tamely ramified of degree $t = (p^2 + 1)(p - 1)$. For $a \neq 0$ we have $\text{ord}_p(a) = \frac{1}{t}$, $\text{ord}_p(b) = \frac{p^2 - p + 1}{t}$, $\text{ord}_p(c) = \frac{p^2}{t}$.*
- iii) *\mathfrak{R}_λ is an \mathbb{F}_{p^4} -vector space under the usual operations in $\overline{\mathbb{W}}/p\overline{\mathbb{W}}$ and $a \mapsto P_a$ defines an $\mathbb{F}_p[G_K]$ -isomorphism $\mathfrak{R}_\lambda \xrightarrow{\sim} E_\lambda$.*

Let $\text{Ext}^1(\mathfrak{E}_\lambda, \mathfrak{E}_\lambda)$ be the group of classes of extensions of finite Honda systems:

$$(3.6) \quad 0 \rightarrow \mathfrak{E}_\lambda \xrightarrow{\iota} (M, L) \xrightarrow{\pi} \mathfrak{E}_\lambda \rightarrow 0$$

under Baer sum. The subgroup $\text{Ext}_{[p]}^1(\mathfrak{E}_\lambda, \mathfrak{E}_\lambda)$ of those classes such that $pM = 0$ was determined in [BK3, Prop. 4.5], as follows.

Proposition 3.7. *If (M, L) represents a class in $\text{Ext}_{[p]}^1(\mathfrak{E}_\lambda, \mathfrak{E}_\lambda)$, then there is a k -basis e_1, \dots, e_8 for M such that $\iota(x_1) = e_1$, $\pi(e_5) = x_1$, $L = \text{span}\{e_1, e_2, e_5, e_6\}$,*

$$V = \left[\begin{array}{cccc|cccc} 0 & 0 & 0 & 0 & 0 & \lambda s_2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & \lambda s_3 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 & \lambda s_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & s_1 & \lambda s_5 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] \text{ and } F = \left[\begin{array}{cccc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -s_1^p & -s_5^p & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & -s_2^p & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{array} \right]$$

with s_1, s_2, s_3, s_4, s_5 in k . For $\tilde{k} = k/(\sigma^4 - 1)(k)$, the map $(M, L) \rightsquigarrow (s_1, \dots, s_5)$ induces an isomorphism of additive groups $\mathbf{s}: \text{Ext}_{[p]}^1(\mathfrak{E}_\lambda, \mathfrak{E}_\lambda) \xrightarrow{\sim} k \oplus k \oplus k \oplus \tilde{k} \oplus k$.

Proposition 3.8 ([BK3, Prop. 5.2.16]). *Let \mathcal{W} be an extension of \mathcal{E}_λ by \mathcal{E}_λ killed by p . The field of points $L = K(\mathcal{W})$ is an elementary abelian p -extension of $F = K(\mathcal{E}_\lambda)$ whose conductor exponent satisfies $\mathfrak{f}(L/F) \leq p^2$.*

4. OUR p -DIVISIBLE GROUPS

We classify Honda systems $(\mathcal{M}, \mathcal{L})$ associated to p -divisible groups whose first layer is \mathcal{E}_α , as in Notation 3.4, with α in k^\times . We also determine the Honda systems of the Cartier duals of such p -divisible groups.

Proposition 4.1. *Let $(\mathcal{M}, \mathcal{L})$ be a Honda system as above. Then there is a basis e_1, e_2, e_3, e_4 for \mathcal{M} over \mathbb{W} and parameters λ in \mathbb{W}^\times and s_1, s_2, s_3, s_5 in \mathbb{W} such that $\lambda \equiv \alpha \pmod{p\mathbb{W}}$, $\mathcal{L} = \text{span}\{e_1, e_2\}$,*

$$V = \left[\begin{array}{cccc} 0 & p\lambda s_2 & 0 & p \\ 1 & p\lambda s_3 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ ps_1 & p\lambda s_5 & p & 0 \end{array} \right] \text{ and } F = \sigma \left[\begin{array}{cccc} 0 & p & -p^2 s_3 & 0 \\ 0 & 0 & p/\lambda & 0 \\ 0 & -ps_1 & p^2 s_1 s_3 - ps_5 & 1 \\ 1 & 0 & -ps_2 & 0 \end{array} \right].$$

Proof. Choose a lift u in \mathbb{W}^\times of α^{-1} and lifts $e_{i,1}$ in \mathcal{M} of a standard basis for \mathfrak{E}_α , such that $e_{4,1} = \text{Fe}_{1,1}$ and $e_{3,1} = \text{Fe}_{4,1}$. We prove by induction that there is a basis $e_{1,n}, \dots, e_{4,n}$ for \mathcal{M} satisfying $e_{4,n} = \text{Fe}_{1,n}$, $e_{3,n} = \text{Fe}_{4,n}$,

$$(4.2) \quad e_{2,n} - Ve_{1,n} \in p^n \text{span}\{e_{3,n}\} + p \text{span}\{e_{4,n}\} \text{ and } e_{3,n} \in uVe_{2,n} + p\mathcal{M}.$$

Substituting the last relation into the first relation in (4.2), we get

$$\begin{aligned} e_{2,n} - Ve_{1,n} &\in p^n(auVe_{2,n} + p\mathcal{M}) + p \text{span}\{e_{4,n}\} \\ &\in p^n auVe_{2,n} + p^{n+1}\mathcal{M} + p \text{span}\{e_{4,n}\}. \end{aligned}$$

Let $e_{1,n+1} = e_{1,n} - \sigma(au)p^n e_{2,n}$. Then

$$(4.3) \quad e_{2,n} - Ve_{1,n+1} \in p^{n+1}\mathcal{M} + p \text{span}\{e_{4,n}\}.$$

Set $e_{4,n+1} = \text{Fe}_{1,n+1}$. We have $\text{Fe}_{2,n} \in p\mathcal{M}$ by Notation 3.4 and $\text{Fe}_{1,n} = e_{4,n}$, so

$$e_{4,n+1} = \text{Fe}_{1,n+1} \text{ is in } \text{Fe}_{1,n} + p^n \text{span}\{\text{Fe}_{2,n}\} \subseteq e_{4,n} + p^{n+1}\mathcal{M}.$$

Hence (4.3) is equivalent to

$$(4.4) \quad e_{2,n} - Ve_{1,n+1} \in p^{n+1}\mathcal{M} + p \text{span}\{e_{4,n+1}\}.$$

Define $e_{3,n+1} = \text{Fe}_{4,n+1}$ and use $\text{Fe}_{4,n} = e_{3,n}$ to obtain

$$e_{3,n+1} = \text{Fe}_{4,n+1} \in \text{Fe}_{4,n} + p^{n+1}\mathcal{M} \subseteq e_{3,n} + p^{n+1}\mathcal{M}.$$

Clearly $\mathcal{M} = \text{span}\{e_{1,n}, e_{2,n}, e_{3,n+1}, e_{4,n+1}\}$. By (4.4), there are scalars b_1, b_2, b_3, b_4 such that

$$e_{2,n} - Ve_{1,n+1} = p^{n+1}\sigma(b_1)e_{1,n} + p^{n+1}\sigma(b_2)e_{2,n} + p^{n+1}b_3e_{3,n+1} + pb_4e_{4,n+1}.$$

Setting $e_{2,n+1} = e_{2,n} - p^{n+1}\sigma(b_1)e_{1,n} - p^{n+1}\sigma(b_2)e_{2,n}$ gives the induction step for the first relation in (4.2). For the second part,

$$e_{3,n+1} \text{ is in } e_{3,n} + p^{n+1}\mathcal{M} \subseteq uVe_{2,n} + p\mathcal{M} \subseteq uVe_{2,n+1} + p\mathcal{M}.$$

Then $e_{1,n}, e_{2,n}, e_{3,n}, e_{4,n}$ converge to a basis e_1, e_2, e_3, e_4 for \mathcal{M} . By the first part of (4.2), $Ve_1 = e_2 + ps_1e_4$ for some s_1 in \mathbb{W} . By the second part, Ve_2 is in $\lambda e_3 + p\text{span}\{e_1, e_2, e_4\}$ for some λ in \mathbb{W}^\times lifting α . Also, $Ve_3 = VFe_4 = pe_4$ and $Ve_4 = VFe_1 = pe_1$. This verifies the matrix for V and that for $F = pV^{-1}$ follows by semi-linearity. \square

Definition 4.5. A basis $\mathcal{B} = \{e_1, e_2, e_3, e_4\}$ as in the Proposition is a *standard basis* for $(\mathcal{M}, \mathcal{L})$. A standard basis for a finite Honda system $(M, L) = (\mathcal{M}/p^n\mathcal{M}, \mathcal{L}/p^n\mathcal{L})$ is the reduction of a standard basis for $(\mathcal{M}, \mathcal{L})$ viewed over $\mathbb{W}/p^n\mathbb{W}$. Denote the associated parameters by $\mathbf{s}_{\mathcal{B}} = [\lambda; s_1, s_2, s_3, s_5]$.

Corollary 4.6. Another basis $\mathcal{B}' = \{e'_i\}$ for $(\mathcal{M}, \mathcal{L})$ is standard if and only if there is an a in \mathbb{W}^\times such that $e'_1 = \sigma^2(a)e_1$, $e'_2 = \sigma(a)e_2$, $e'_3 = \sigma^4(a)e_3$ and $e'_4 = \sigma^3(a)e_4$. Then $\mathbf{s}_{\mathcal{B}'}$ is given by

$$\lambda' = \frac{a}{\sigma^4(a)}\lambda, \quad s'_1 = \frac{\sigma(a)}{\sigma^3(a)}s_1, \quad s'_2 = \frac{\sigma^4(a)}{\sigma^2(a)}s_2, \quad s'_3 = \frac{\sigma^4(a)}{\sigma(a)}s_3 \quad \text{and} \quad s'_5 = \frac{\sigma^4(a)}{\sigma^3(a)}s_5.$$

Proof. Since $\mathcal{L} = \text{span}\{e_1, e_2\} = \text{span}\{e'_1, e'_2\}$, we can find a, b, c, d in \mathbb{W} such that

$$e'_1 = \sigma^2(a)e_1 + \sigma(b)e_2 \quad \text{and} \quad e'_2 = ce_1 + \sigma(d)e_2.$$

Then $e'_4 = Fe'_1 = \sigma^3(a)e_4 + \sigma^2(b)(pe_1 - p\sigma(s_1)e_3)$ and

$$\begin{aligned} Ve'_1 &= \sigma(a)V(e_1) + bV(e_2) \\ &= \sigma(a)(e_2 + ps_1e_4) + b(p\lambda s_2e_1 + p\lambda s_3e_2 + \lambda e_3 + p\lambda s_5e_4). \end{aligned}$$

From the matrix for V on the new basis, we have:

$$Ve'_1 = e'_2 + ps'_1e'_4 = ce_1 + \sigma(d)e_2 + ps'_1(\sigma^3(a)e_4 + \sigma^2(b)(pe_1 - p\sigma(s_1)e_3)).$$

Comparing coefficients of e_3 in Ve'_1 gives $\lambda b = -p^2s'_1\sigma(s_1)\sigma^2(b)$, so $b = 0$ or else $\text{ord}_p(b) \geq 2 + \text{ord}_p(b)$. By comparing coefficients of e_1 , we find that $c = 0$. Hence the coefficients of e_2 and e_4 give $d = a$ and $s'_1\sigma^3(a) = \sigma(a)s_1$. We now have:

$$e'_1 = \sigma^2(a)e_1, \quad e'_2 = \sigma(a)e_2, \quad e'_4 = \sigma^3(a)e_4 \quad \text{and} \quad e'_3 = Fe'_4 = \sigma^4(a)e_3.$$

Compare coefficients in $Ve'_2 = aV(e_2) = a\lambda(ps_2e_1 + ps_3e_2 + e_3 + ps_5e_4)$ and

$$\begin{aligned} Ve'_2 &= \lambda'(ps'_2e'_1 + ps'_3e'_2 + e'_3 + ps_5e'_4) \\ &= \lambda'(ps'_2\sigma^2(a)e_1 + ps'_3\sigma(a)e_2 + \sigma^4(a)e_3 + ps'_5\sigma^3(a)e_4). \quad \square \end{aligned}$$

Corollary 4.7. Let $(\mathcal{M}^*, \mathcal{L}^*)$ be the dual of $(\mathcal{M}, \mathcal{L})$ as in Lemma 3.2. There is a standard basis $\tilde{\mathcal{B}} = \{\xi_i\}$ for \mathcal{M}^* with $\mathbf{s}_{\tilde{\mathcal{B}}} = [\lambda'; s'_1, s'_2, s'_3, s'_5]$ related to $\mathbf{s}_{\mathcal{B}}$ by:

$$\begin{aligned} \lambda' &= \frac{1}{\sigma^2(\lambda)}, \quad s'_1 = -\frac{1}{\sigma(\lambda)}s_1, \quad s'_2 = -\sigma^2(\lambda)s_2, \quad s'_3 = \sigma^2(\lambda)\sigma^{-1}(ps_1s_3 - s_5), \\ s'_5 &= -\sigma^2(\lambda)\sigma(s_3) - \frac{ps_1\sigma^2(\lambda)}{\sigma(\lambda)}\sigma^{-1}(ps_1s_3 - s_5). \end{aligned}$$

Proof. By (3.3) and the Proposition, the matrices for Verschiebung and Frobenius on \mathcal{M}^* in terms of its dual basis e_1^*, \dots, e_4^* are given by

$$V = \begin{bmatrix} 0 & 0 & 0 & 1 \\ p & 0 & -ps_1 & 0 \\ -p^2s_3 & p/\lambda & p^2s_1s_3 - ps_5 & -ps_2 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad F = \sigma \begin{bmatrix} 0 & 1 & 0 & ps_1 \\ p\lambda s_2 & p\lambda s_3 & \lambda & p\lambda s_5 \\ 0 & 0 & 0 & p \\ p & 0 & 0 & 0 \end{bmatrix}.$$

Since \mathcal{L}^* is the annihilator of \mathcal{L} , we have $\mathcal{L}^* = \text{span}\{e_3^*, e_4^*\}$. For the standard basis, ξ_2 is in \mathcal{L}^* while $F\xi_2$ is in $p\mathcal{M}^*$, so $\xi_2 = xe_4^* + pwe_3^*$ with x in \mathbb{W}^\times and w in \mathbb{W} . But $\{\xi_1, \xi_2\}$ is a basis for \mathcal{L}^* , so $\xi_1 = ye_3^* + ze_4^*$, with y in \mathbb{W}^\times . By scaling, assume that $y = 1$ and $\xi_1 = e_3^* + ze_4^*$. Apply V and let $t_5 = ps_1s_3 - s_5$, to find that

$$V(\xi_1) = \sigma^{-1}(z)e_1^* - ps_1e_2^* + p[t_5 - \sigma^{-1}(z)s_2]e_3^* + e_4^*.$$

Proposition 4.1 gives F and V on the standard basis $\tilde{\mathcal{B}}$ for \mathcal{M}^* . Thus:

$$\xi_4 = F(\xi_1) = p\sigma(s_1z)e_1^* + \sigma(\lambda + p\lambda s_5z)e_2^* + p\sigma(z)e_3^*,$$

$$V(\xi_1) = \xi_2 + ps_1'\xi_4 = xe_4^* + pwe_3^* + ps_1'(p\sigma(s_1z)e_1^* + \sigma(\lambda + p\lambda s_5z)e_2^* + p\sigma(z)e_3^*).$$

Equating the coefficient of e_1^* gives $\sigma^{-1}(z) = ps_1'\sigma(s_1z)$, whose valuation implies that $z = 0$. Comparing the other coefficients gives $x = 1$, $w = t_5$ and $\sigma(\lambda)s_1' = -s_1$, so $\xi_1 = e_3^*$, $\xi_2 = e_4^* + pt_5e_3^*$, $\xi_4 = \sigma(\lambda)e_2^*$, $\xi_3 = F\xi_4 = \sigma^2(\lambda)(e_1^* + p\sigma(\lambda s_3)e_2^*)$. The remaining formulas relating $\mathbf{s}_{\tilde{\mathcal{B}}}$ and $\mathbf{s}_{\mathcal{B}}$ result from a comparison of

$$\begin{aligned} V(\xi_2) &= V(e_4^*) + p\sigma^{-1}(t_5)V(e_3^*) \\ &= e_1^* - ps_2e_3^* + p\sigma^{-1}(t_5)(-ps_1e_2^* + pt_5e_3^* + e_4^*) \\ &= e_1^* - p^2s_1\sigma^{-1}(t_5)e_2^* - p(s_2 - p\sigma^{-1}(t_5)t_5)e_3^* + p\sigma^{-1}(t_5)e_4^* \end{aligned}$$

with

$$\begin{aligned} V(\xi_2) &= \lambda' (ps_2'\xi_1 + ps_3'\xi_2 + \xi_3 + ps_5'\xi_4) \\ &= \lambda' (ps_2'e_3^* + ps_3'(e_4^* + pt_5e_3^*) + \sigma^2(\lambda)(e_1^* + p\sigma(\lambda s_3)e_2^*) + ps_5'\sigma(\lambda)e_2^*) \\ &= \lambda' (\sigma^2(\lambda)e_1^* + p(\sigma^2(\lambda)\sigma(\lambda s_3) + \sigma(\lambda)s_5')e_2^* + p(s_2' + pt_5s_3')e_3^* + ps_5'e_4^*). \end{aligned}$$

Corollary 4.8. *Let \mathcal{B} be a standard basis for \mathcal{M} and $\mathbf{s}_{\mathcal{B}} = [\lambda; s_1, s_2, s_3, s_5]$. Then $(\mathcal{M}^*, \mathcal{L}^*)$ and $(\mathcal{M}, \mathcal{L})$ are isomorphic if and only if there are a, b in \mathbb{W}^\times such that*

$$\lambda = -\frac{\sigma^2(a)}{a}b, \quad s_5 = \frac{\sigma^3(a)}{\sigma^2(a)}\sigma(bs_3) + ps_1s_3$$

and one of the following holds: i) $s_1 \neq 0$ (or $s_2 \neq 0$) and $b = 1$; ii) $s_1 = s_2 = 0$ and $b = \pm 1$; iii) all $s_j = 0$ and $b\sigma^2(b) = 1$.

Proof. Assume that $(\mathcal{M}, \mathcal{L})$ and $(\mathcal{M}^*, \mathcal{L}^*)$ are isomorphic and use the previous Corollaries for the relationship between $\mathbf{s}_{\tilde{\mathcal{B}}}$ and $\mathbf{s}_{\mathcal{B}}$. In particular, $\lambda\sigma^2(\lambda) = \frac{\sigma^4(a)}{a}$ for some a in \mathbb{W}^\times . Define b in \mathbb{W}^\times by $\lambda = -\frac{\sigma^2(a)}{a}b$. Then $b\sigma^2(b) = 1$ and the requirement on s_3' implies our claimed formula for s_5 . In case (i), the condition on s_1' (or s_2') forces $b = 1$. In case (ii), use s_3' to find that $b = \pm 1$. Conversely, in each of these cases, there is an isomorphism. \square

Example 4.9. Since the finite Honda system with parameters $\mathbf{s}_{\mathcal{B}} = [\lambda; 0, 0, 0, 0]$ plays an important role in later conductor estimates, note that it occurs naturally for $p = 2$. Let $A = J(C)$ be the Jacobian of the curve $C: y^2 + ay = x^5 + b$ with a and b units in \mathbb{W} . Then $A[2]$ is isomorphic to \mathcal{E}_λ as in Notation 3.4. Without loss of generality, we may assume that a primitive fifth root of unity ζ_5 is in \mathbb{W} . The

automorphism $(x, y) \mapsto (\zeta_5 x, y)$, induces a complex multiplication on A and so an isomorphism φ of the associated Honda system $(\mathcal{M}, \mathcal{L})$. If \mathcal{B} is a standard basis, so is $\mathcal{B}' = \varphi(\mathcal{B})$ and $\mathbf{s}_{\mathcal{B}} = [\lambda; s_1, s_2, s_3, s_5]$ is related to $\mathbf{s}_{\mathcal{B}'} = [\lambda'; s'_1, s'_2, s'_3, s'_5]$ as in Corollary 4.6. Moreover, $\varphi(e_1) = \sigma^2(\zeta)e_1$, where ζ is a primitive fifth root of unity, since φ has order 5 on \mathcal{M} . Because φ commutes with V , each basis leads to the same matrix for V , as in Proposition 4.1 and thus $\mathbf{s}_{\mathcal{B}'} = \mathbf{s}_{\mathcal{B}}$. But s'_i is a multiple of s_i by a primitive fifth root of unity, so each $s_i = 0$.

5. EXPONENT p^2

Let $(\mathcal{M}, \mathcal{L})$ be the Honda system of a p -divisible group as in Proposition 4.1. Throughout this section, $\mathfrak{M} = (M, L)$ denotes the finite Honda system of exponent p^2 such that $(M, L) \simeq (\mathcal{M}/p^2\mathcal{M}, \mathcal{L}/p^2\mathcal{L})$, with standard basis $\mathcal{B} = \{e_1, e_2, e_3, e_4\}$ over $\mathbb{W}/p^2\mathbb{W}$ and parameters $\mathbf{s}_{\mathcal{B}}(\mathfrak{M}) = [\lambda; s_1, s_2, s_3, s_5]$, where λ is in $(\mathbb{W}/p^2\mathbb{W})^\times$ and the s_i are in $k = \mathbb{W}/p\mathbb{W}$. Write \mathcal{V} for the group scheme associated to \mathfrak{M} .

5.1. Baer sums for exponent p^2 . Let x_1, \dots, x_4 be a standard basis for \mathfrak{E}_λ , cf. Notation 3.4. Then \mathfrak{M} is an extension of $\mathfrak{E}_\lambda \simeq (M/pM, L/pL)$ by $\mathfrak{E}_\lambda \simeq (pM, pL)$:

$$(5.1.1) \quad 0 \rightarrow \mathfrak{E}_\lambda \xrightarrow{\iota} (M, L) \xrightarrow{\pi} \mathfrak{E}_\lambda \rightarrow 0,$$

with D_k -maps ι and π induced by $\iota(x_1) = pe_1$ and $\pi(e_1) = x_1$. Write $[\mathfrak{M}]$ for the class in $\text{Ext}^1(\mathfrak{E}_\lambda, \mathfrak{E}_\lambda)$ of the extension (5.1.1). Let $\mathfrak{M}' = (M', L')$ be the finite Honda system of Proposition 3.7 with $pM' = 0$ and parameters $\lambda \bmod p$ in k^\times and $\mathbf{s}(\mathfrak{M}') = (s'_1, s_2, s'_3, 0, s'_5)$ in k^5 with respect to a standard basis e'_1, \dots, e'_8 .

Proposition 5.1.2. *The Baer sum $[\mathfrak{M}] + [\mathfrak{M}']$ in $\text{Ext}^1(\mathfrak{E}_\lambda, \mathfrak{E}_\lambda)$ is the extension class of $\mathfrak{M}'' = (M'', L'')$, where M'' has exponent p^2 and its parameters with respect to a standard basis are $\mathbf{s}_{\mathcal{B}''}(\mathfrak{M}'') = [\lambda; s_1 + s'_1, s_2 + s'_2, s_3 + s'_3, s_5 + s'_5]$.*

Proof. To construct \mathfrak{M}'' , let $\Gamma = \{(m, m') \in M \oplus M' \mid \pi(m) = \pi'(m')\}$ be the fiber product with coordinatewise action of the Dieudonné ring D_k . Impose the relations $\Delta = \{(\iota(x), 0) - (0, \iota'(x)) \in \Gamma \mid x \in \mathfrak{E}_\lambda\}$, so that $M'' = \Gamma/\Delta$ and L'' is the image in M'' of $\{(p, p') \in L \oplus L' \mid \pi(p) = \pi'(p')\}$. Then $\mathfrak{M}'' = (M'', L'')$ is an extension:

$$0 \rightarrow \mathfrak{E}_\lambda \xrightarrow{\iota''} (M'', L'') \xrightarrow{\pi''} \mathfrak{E}_\lambda \rightarrow 0,$$

with ι'' induced by $x \rightsquigarrow (x, 0)$ and π'' by $(m, m') \rightsquigarrow \pi(m)$. The elements

$$\gamma_1 = (e_1, e'_5), \quad \gamma_2 = (e_2, e'_6), \quad \gamma_3 = (e_3, e'_7), \quad \gamma_4 = (e_4, e'_8)$$

of $M \oplus M'$ satisfy the fiber product condition for membership in Γ and we claim that their cosets modulo Δ form a standard basis for M'' over $\mathbb{W}/p^2\mathbb{W}$. Indeed, $L'' = \text{span}\{\gamma_1, \gamma_2\}$, $F\gamma_1 = \gamma_4$ and $F\gamma_4 = \gamma_3$.

Write $(m_1, m'_1) \sim (m_2, m'_2)$ if each pair represents the same coset in M'' . The relations on Γ give $(0, e'_i) \sim (pe_i, 0)$ for $1 \leq i \leq 4$, so

$$V\gamma_1 = (e_2 + ps_1e_4, e'_6 + s'_1e'_4) = (e_2, e'_6) + (p(s_1 + s'_1)e_4, 0) \sim \gamma_2 + p(s_1 + s'_1)\gamma_4.$$

A similar computation, using $s'_4 = 0$, shows that

$$(5.1.3) \quad V\gamma_2 \sim \lambda(\gamma_3 + p(s_2 + s'_2)\gamma_1 + p(s_3 + s'_3)\gamma_2 + p(s_5 + s'_5)\gamma_4).$$

Complete the remaining entries of V and F to verify that $\gamma_1, \dots, \gamma_4$ is a standard basis for \mathfrak{M}'' . Then the parameters of \mathfrak{M}'' can be read off (5.1.3). \square

5.2. Field of points and conductor for exponent p^2 . We first determine the field of points for the group scheme $\mathcal{V}^{(0)}$ associated to the finite Honda system $\mathfrak{M}^{(0)}$ of exponent p^2 with parameters $[\lambda; 0, 0, 0, 0]$.

Lemma 5.2.1. *The field $L_0 = K(\mathcal{V}^{(0)})$ is the compositum of $F = K(\mathcal{E}_\lambda)$ and $\mathbb{Q}_p(z)$, where z is any root of*

$$(5.2.2) \quad f(Z) = \left(Z^{p^3} + \frac{1}{p}\right)^p + (-1)^p p^{p-1} Z^{p^4} - Z - \delta_p$$

with $\delta_p = 1$ if $p = 2$ and 0 otherwise.

Proof. Points of $\mathcal{V}^{(0)}$ correspond to D_k -homomorphisms $\varphi: M \rightarrow \widehat{CW}_k(\overline{\mathbb{W}}/p\overline{\mathbb{W}})$. Let e_1, e_2, e_3, e_4 be a standard basis for $\mathfrak{M}^{(0)} = (M, L)$. Then e_1 generates M as a D_k -module and V^4 vanishes on M by Proposition 4.1. Hence φ is determined by $\varphi(e_1) = (\vec{0}, y_3, y_2, y_1, y_0)$. By applying F to e_1 and e_4 , we find that:

$$(5.2.3) \quad \varphi(e_4) = (\vec{0}, y_3^p, y_2^p, y_1^p, y_0^p) \quad \text{and} \quad \varphi(e_3) = (\vec{0}, y_3^{p^2}, y_2^{p^2}, y_1^{p^2}, y_0^{p^2}).$$

Since $\{f_i = pe_i \mid 1 \leq i \leq 4\}$ is a standard basis for $(pM, pL) \simeq \mathfrak{E}_\lambda$, we have $\varphi(f_1) = (\vec{0}, c, b, a)$, where a is a root of $\lambda^{p^2} a^{p^4-1} = (-p)^{p+1}$ as in Proposition 3.5. Then

$$(\vec{0}, y_3^p, y_2^p, y_1^p) = VF\varphi(e_1) = \varphi(f_1) = (\vec{0}, c, b, a)$$

and so

$$y_1^p \equiv a, \quad y_2^p \equiv b \equiv -\frac{1}{p}\lambda^p a^{p^3}, \quad y_3^p \equiv c \equiv \lambda a^{p^2} \pmod{p\overline{\mathbb{W}}}.$$

Also, $\varphi(e_2) = V\varphi(e_1) = (\vec{0}, y_3, y_2, y_1)$. Use Proposition 3.5(ii) to evaluate:

$$\text{ord}_p(y_1) = \frac{1}{p} \text{ord}_p(a), \quad \text{ord}_p(y_2) = \frac{1}{p} \text{ord}_p(b), \quad \text{ord}_p(y_3) = \frac{1}{p} \text{ord}_p(c).$$

With these estimates, (5.2.3) gives $\varphi(e_3) = (\vec{0}, a^p, y_0^{p^2})$. From $V(e_2) = \lambda e_3$, we have:

$$(\vec{0}, y_3, y_2) = V\varphi(e_2) = \lambda\varphi(e_3) = \lambda(\vec{0}, a^p, y_0^{p^2}) = (\vec{0}, \lambda^{1/p} a^p, \lambda y_0^{p^2}).$$

Thus $y_3 \equiv \lambda^{1/p} a^p$ and $y_2 \equiv \lambda y_0^{p^2} \pmod{p\overline{\mathbb{W}}}$.

Vanishing of the Hasse-Witt exponential map on L implies that:

$$y_0 + \frac{y_1^p}{p} + \frac{y_2^{p^2}}{p^2} + \frac{y_3^{p^3}}{p^3} \equiv 0 \quad \text{and} \quad y_1 + \frac{y_2^p}{p} + \frac{y_3^{p^2}}{p^2} \equiv 0 \pmod{p\overline{\mathbb{W}}}$$

and so

$$y_0 + (-1)^p \frac{\lambda^{p^2}}{p} \left(\frac{y_0^{p^3}}{p} + \frac{a^{p^3}}{p^2} \right)^p + \frac{\lambda^{p^2}}{p^2} y_0^{p^4} + \delta_p a \equiv 0 \pmod{p\overline{\mathbb{W}}}.$$

The substitution $y_0 = az$ relates y_0 to a root z in $\overline{\mathbb{W}}$ of (5.2.2). Since $V^{(0)}$ is generated by the points of E_λ and a choice of φ , we have $L_0 = F(z)$. \square

Notation 5.2.4. If x is an element of a p -adic ring R , the *big-0* notation $0(x)$ represents an element of the ideal (x) .

Lemma 5.2.5. *Let r be a root of $x^{p^3} - x^{p^2} + x^p - x + \frac{1}{p}$. Then:*

- i) $[\mathbb{Q}_p(r):\mathbb{Q}_p] = p^3$, a prime of $\mathbb{Q}_p(r)$ is given by $\pi = \frac{1}{r}$ and $\text{ord}_p(r) = -\frac{1}{p^3}$.
- ii) The splitting field of $f(Z)$ in (5.2.2) is $\mathbb{Q}_p(\mu_{p^4-1}, r)$.
- iii) The roots of $f(Z)$ have the form $r + \alpha + 0(\pi^{(p-1)p})$ with α in $\{0\} \cup \mu_{p^4-1}$.

Proof. Since $\text{ord}_p(r) = -\frac{1}{p^3}$, the equation for r is irreducible over \mathbb{Q}_p and (i) holds. For convenience, let $\epsilon = \pi^{p^2-p}$ in $\mathbb{Q}_p(r)$, so $\text{ord}_p(\epsilon) = \frac{p-1}{p^2}$. First we show that r is an approximate root of f satisfying $f(r) = 0(\epsilon)$.

Let $s = r^{p^3} + \frac{1}{p} = r^{p^2} - r^p + r$, so $\text{ord}_p(s) = -\frac{1}{p}$. By binominal expansion:

$$s^p = (r^{p^2} - r^p + r)^p = r^{p^3} + (-1)^p r^{p^2} + r^p + 0(\epsilon) = r - \frac{1}{p} + 0(\epsilon)$$

since the worst case middle term satisfies $\text{ord}_p(p r^{p^2(p-1)} r^p) = \frac{p-1}{p^2}$ and also using $\text{ord}_2(2r^4) = \frac{1}{2} > \text{ord}_2(\epsilon)$ when $p = 2$. Similarly,

$$(-1)^p p^{p-1} r^{p^4} = (-1)^p \frac{1}{p} (p r^{p^3})^p = \frac{1}{p} (1 - ps)^p = \frac{1}{p} + (-1)^p p^{p-1} s^p + \epsilon_1$$

with $\text{ord}_p(\epsilon_1) = \text{ord}_p(ps) = \frac{p-1}{p} > \text{ord}_p(\epsilon)$. By the formula for s^p above, we have

$$(-1)^p p^{p-1} r^{p^4} = \frac{1}{p} + (-1)^p p^{p-1} (r - \frac{1}{p} + 0(\epsilon)) + \epsilon_1 = \frac{1}{p} + (-1)^{p+1} p^{p-2} + 0(\epsilon)$$

since $\text{ord}_p(p^{p-1} r) \geq 1 - \frac{1}{p^3}$. Modulo p , replace $(-1)^{p+1} p^{p-2}$ with δ_p . Then

$$\begin{aligned} f(r) &= s^p + (-1)^p p^{p-1} r^{p^4} - r - \delta_p \\ &= \left(r - \frac{1}{p}\right) + \left(\frac{1}{p} + \delta_p\right) - r - \delta_p + 0(\epsilon) = 0(\epsilon). \end{aligned}$$

We next show that the coefficients of $g(y) = f(y+r) - f(r)$ lie in $\mathbb{Z}_p[\pi]$ and that $g(y) \equiv y^{p^4} - y \pmod{\pi^2}$. If we expand

$$\begin{aligned} \left((y+r)^{p^3} + \frac{1}{p}\right)^p &= y^{p^4} + \left(r^{p^3} + \frac{1}{p}\right)^p + h_1(y) \quad \text{and} \\ p^{p-1}(y+r)^{p^4} &= p^{p-1} r^{p^4} + h_2(y), \end{aligned}$$

then $g(y) = y^{p^4} - y + h_1(y) + (-1)^p h_2(y)$. Let $C_j(h)$ denote the coefficient of y^j in any polynomial $h(y)$ and recall that $\text{ord}_p\left(\binom{p^n}{j}\right) = n - \text{ord}_p(j)$. The terms in $h_2(y)$ involve y^j for $1 \leq j \leq p^4$ and we have

$$(5.2.6) \quad \text{ord}_p(C_j(h_2)) = p - 1 + \text{ord}_p\left(\left(\binom{p^4}{j}\right) r^{p^4-j}\right) = 3 - \text{ord}_p(j) + \frac{j}{p^3} \geq 1.$$

(The minimum occurs for $j = p^3$; also for $j = 16$ if $p = 2$.) Thus $h_2(y) \equiv 0 \pmod{p}$. To estimate $\text{ord}_p(C_j(h_1))$, recall that $s = r^{p^3} + \frac{1}{p}$ and $\text{ord}_p(s) = -\frac{1}{p}$, so

$$(5.2.7) \quad \text{ord}_p\left(\binom{p}{i} s^i\right) = 1 - \frac{i}{p} \geq \frac{1}{p} \quad \text{for } 1 \leq i \leq p-1.$$

Similarly, $(y+r)^{p^3} = y^{p^3} + r^{p^3} + h_3(y)$, where h_3 involves y^j for $1 \leq j \leq p^3 - 1$, and we have

$$\text{ord}_p(C_j(h_3)) = \text{ord}_p\left(\left(\binom{p^3}{j}\right) r^{p^3-j}\right) = 2 - \text{ord}_p(j) + \frac{j}{p^3} \geq \frac{1}{p}.$$

(The minimum occurs when $j = p^2$.) It follows that

$$(5.2.8) \quad (y^{p^3} + h_3(y))^p \equiv y^{p^4} \pmod{p}.$$

By definition of h_1 , we now find that

$$\begin{aligned} y^{p^4} + s^p + h_1(y) &= ((y+r)^{p^3} + \frac{1}{p})^p = (y^{p^3} + r^{p^3} + h_3(y) + \frac{1}{p})^p \\ &= (y^{p^3} + h_3(y) + s)^p = (y^{p^3} + h_3(y))^p + s^p + h_4(y) \end{aligned}$$

with $\text{ord}_p(C_j(h_4)) \geq \frac{1}{p}$ by (5.2.7). Moreover, $h_1(y) \equiv h_4(y) \pmod{p}$ by (5.2.8). Thus $g(y) \equiv y^{p^4} - y \pmod{\pi^{p^2}}$.

Modulo $\pi^{(p-1)p}$, we have $f(r) \equiv 0$ and thus $f(y+r) = f(r) + g(y) \equiv y^{p^4} - y$. If α is in $\{0\} \cup \mu_{p^4-1}$, then $f(r+\alpha) \equiv 0 \pmod{\pi^{(p-1)p}}$ and $f'(r+\alpha)$ is a unit. We conclude by Hensel's Lemma that there is a root θ_α of f in $\mathbb{Z}_p[r]$ such that $\text{ord}_p(\theta_\alpha - (r+\alpha)) \geq \frac{p-1}{p}$ and this account for all the roots of f . \square

Proposition 5.2.9. *The field of points $L_0 = F(\mathcal{V}^{(0)})$ is an elementary abelian p -extension of $F = K(E_\lambda)$, totally ramified of degree p^3 , with ray class conductor exponent $\mathfrak{f}(L_0/F) = p^3 - p^2 + p$.*

Proof. Since $H = \text{Gal}(L_0/F)$ is trivial on E_λ , it is an elementary abelian p -group. By the Lemmas above, $L_0 = F(z) = F(r)$ because F contains μ_{p^4-1} . But the ramification in F/\mathbb{Q}_p is tame of degree $t = (p-1)(p^2+1)$ by Proposition 3.5, while $\mathbb{Q}_p(r)/\mathbb{Q}_p$ is totally ramified of degree p^3 by Lemma 5.2.5, so L_0/F also is totally ramified of degree p^3 . With respect to a prime element π' of L_0 , we have $\text{ord}_{\pi'}(\frac{1}{r}) = p^3 t \text{ord}_p(\frac{1}{r}) = t$. Moreover $h(r) - r$ is a unit for all $h \neq 1$ in H . By a conductor lemma [BK3, A.5], we find that $\mathfrak{f}(L_0/F) = t+1 = p^3 - p^2 + p$. \square

Proposition 5.2.10. *Let \mathcal{V} be the group scheme associated to a finite Honda system $\mathfrak{M} = (\mathcal{M}/p^2\mathcal{M}, \mathcal{L}/p^2\mathcal{L})$ with $(\mathcal{M}, \mathcal{L})$ as in Proposition 4.1. Then $L = K(\mathcal{V})$ is an elementary abelian p -extension of $F = K(E_\lambda)$ with $\mathfrak{f}(L/F) = p^3 - p^2 + p$.*

Proof. Denote the parameters of \mathfrak{M} by $\mathbf{s} = \mathbf{s}_B(\mathfrak{M}) = [\lambda; s_1, s_2, s_3, s_5]$. Proposition 5.1.2 shows that there is a finite Honda system \mathfrak{M}' of exponent p with parameters $\lambda \pmod{p}$ and $\mathbf{s}(\mathfrak{M}') = (s_1, s_2, s_3, 0, s_5)$ such that the Baer sum of extension classes satisfies $[\mathfrak{M}] = [\mathfrak{M}^{(0)}] + [\mathfrak{M}']$. If $L' = K(\mathcal{V}')$ is the field of points of the group scheme \mathcal{V}' associated to \mathfrak{M}' , it follows from the fiber product construction that $L \subseteq L_0 L'$. The respective conductor exponents $\mathfrak{f} = \mathfrak{f}(L/F)$, $\mathfrak{f}_0 = \mathfrak{f}(L_0/F) = p^3 - p^2 + p$ and $\mathfrak{f}' = \mathfrak{f}(L'/F)$ satisfy $\mathfrak{f} \leq \max\{\mathfrak{f}_0, \mathfrak{f}'\}$, cf. [BK3, Lemma A.9]. Proposition 3.8 asserts that $\mathfrak{f}' \leq p^2$ and so $\mathfrak{f} \leq \mathfrak{f}_0$. But also, $[\mathfrak{M}^{(0)}] = [\mathfrak{M}] + (-[\mathfrak{M}'])$, so $\mathfrak{f}_0 \leq \mathfrak{f}$. \square

Remark 5.2.11. In contrast, Fontaine's bound on higher ramification implies that $\mathfrak{f}(L/F) \leq p^3 + p + 1$, cf. [BK3, Prop. A.11]. The sharper bound in Proposition 5.2.10 is essential for our applications. In particular, when $p = 2$, we find that $\sqrt{2}$ is not in L , since $\mathfrak{f}(F(\sqrt{2})/F) = 11$.

5.3. Finding the Honda parameters for $A[4]$. While it seems difficult to compute Honda parameters in general, we have the following explicit result.

Proposition 5.3.1. *Let $g(x) = a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ and let C be the curve $y^2 + y = g(x)$ over \mathbb{Z}_2 with a_5 a unit. If $A = J(C)$ is the Jacobian of C , the Honda parameters associated to $A[4]$ are given by $\lambda \equiv -1 \pmod{4}$ and*

$$s_1 \equiv a_1 + a_3a_4 + \frac{a_3^2 - a_3}{2}, \quad s_2 \equiv a_3, \quad s_3 \equiv s_5 \equiv a_1 + a_2 + a_3 + a_4 \pmod{2}.$$

Proof. We sketch the argument, invoking several lemmas to be proved below. Consider a deformation $\tilde{A} = J(\tilde{C})$, where \tilde{C} is the curve $y^2 + y = \tilde{g}(x)$ with

$$(5.3.2) \quad \tilde{g} = a_5x^5 + b_3x^3 + b_2x^2 + b_1x + b_0.$$

In Lemma 5.3.8, we prove an effective deformation result under which $\tilde{A}[4]$ and $A[4]$ share the same Honda parameters, leaving 32 explicit curves over \mathbb{Z}_2 for which

the Honda parameters must be determined. Lemma 5.3.4 gives the Kummer group associated to $F(\frac{1}{2}P)$ for a point P of order 2 on A . A comparison with the Kummer group coming from the finite Honda system in §5.2, as sketched at the end of this subsection, verifies our claim \square

Remark 5.3.3. Note that the eight Honda parameters \mathbf{s}_B for $A[4]$ over \mathbb{Z}_2 are consistent with the duality in Corollary 4.8.

For the lemmas needed above, let K be a field not of characteristic 2. Recall the $x - T$ map ([Sch], [CF, Ch. 6], [PS, §5]), which gives an explicit interpretation of the Kummer-theoretic boundary map $A(K)/2A(K) \hookrightarrow H^1(G_K, A[2])$ arising from Galois cohomology of

$$0 \rightarrow A[2] \rightarrow A(\overline{K}) \xrightarrow{2} A(\overline{K}) \rightarrow 0.$$

Let $A = J(C)$ be the Jacobian of the genus 2 curve $C: y^2 = f(x)$ over K . As is well-known [CF, Ch. 2], the effective divisor \mathfrak{D} of degree 2 on C lying over $x = \infty$ represents the canonical divisor class and every K -rational divisor class of degree 0 is represented by a unique divisor of the form $\mathfrak{A} - \mathfrak{D}$, with \mathfrak{A} effective of degree 2 and defined over K . For our applications, assume that $\deg f = 6$ and factor

$$f(x) = c(x - r_1) \cdots (x - r_6)$$

over the splitting field F of f . Let \mathcal{H}_F be the quotient of $\bigoplus_{j=1}^6 F^\times / F^{\times 2}$ by the image of F^\times on the diagonal. If $\mathfrak{A} = P_1 + P_2$, where the points $P_i = (x_i, y_i)$ are affine and x_i is not a root of f , then the j -th coordinate of the $x - T$ homomorphism $\partial_F: A(F) \rightarrow \mathcal{H}_F$ is induced by $\mathfrak{A} - \mathfrak{D} \rightsquigarrow (x_1 - r_j)(x_2 - r_j)$. However, if say $x_1 = r_j$, then a non-zero entry is obtained by replacing $x_1 - r_j$ by $f'(r_j)$, since

$$\frac{y^2}{x - r_j} = \frac{f(x)}{x - r_j} = c \prod_{\substack{i=1, \dots, 6 \\ i \neq j}} (x - r_i).$$

The kernel of ∂_F is $2A[F]$. If $\partial_F(P)$ is represented by (c_1, \dots, c_6) in $\bigoplus F^\times$, then the product $c_1 \cdots c_6$ is in $F^{\times 2}$.

Lemma 5.3.4. *Let P in $A[2]$ be represented by the divisor $(r_5, 0) + (r_6, 0) - \mathfrak{D}$ and let $q(x) = (x - r_5)(x - r_6)$. Then $\partial_F(P)$ is represented by*

$$(q(r_1), q(r_2), q(r_3), q(r_4), (r_6 - r_5)f'(r_5), (r_5 - r_6)f'(r_6)).$$

The elementary 2-extension $F(\frac{1}{2}P)/F$ is generated by the square roots of

$$\frac{c(r_5 - r_1)(r_6 - r_2)}{(r_6 - r_3)(r_6 - r_4)}, \frac{c(r_5 - r_2)(r_6 - r_1)}{(r_6 - r_3)(r_6 - r_4)}, \frac{c(r_5 - r_3)(r_6 - r_1)}{(r_6 - r_2)(r_6 - r_4)}, \frac{c(r_5 - r_4)(r_6 - r_1)}{(r_6 - r_2)(r_6 - r_3)}.$$

Proof. We find $\partial_F(P)$ by the definition reviewed above. Since the diagonal image of F^\times is trivial in \mathcal{H}_K , we divide by the last coordinate. Then the product of the first five coordinates is a square, so adjoining the square roots of the first four coordinates is necessary and sufficient to obtain $F(\frac{1}{2}P)$. \square

Now let k be a finite field of characteristic 2, let K be the field of fractions of $\mathbb{W} = \mathbb{W}(k)$ and let $A = J(C)$ as in Proposition 5.3.1. Replace x by a scalar multiple to arrange that a_5 be one of (at most 5) fixed representatives for $\mathbb{W}^\times / \mathbb{W}^{\times 5}$ and

translate to make the coefficient of x^4 zero. Then C has a model of the form $y^2 = x\Phi(x)$, where $\Phi(x)$ is the quintic

$$(5.3.5) \quad \Phi(x) = x^5(1 + 4g(1/x)) = (4a_0 + 1)x^5 + 4a_1x^4 + 4a_2x^3 + 4a_3x^2 + 4a_5.$$

Let $r_6 = 0$ and let r_1, \dots, r_5 be the roots of Φ .

Lemma 5.3.6. *The 2-division field F of A is $K(\zeta, \pi)$, where ζ is a primitive fifth root of unity and π is a root of $a_5^2z^5 - 2$. The roots of Φ have the form*

$$r_j = -a_5\zeta^j\pi^2 + 0(2\pi), \quad j = 1, \dots, 5.$$

Proof. Let $L = K(\zeta, \pi)$ and $d(z) = \frac{1}{\pi^{10}}\Phi(\pi^2z) \equiv z^5 + a_5^5 \pmod{\pi^4\mathcal{O}_L[z]}$. By Hensel's Lemma, the approximate root $-a_5\zeta^j$ leads to a root of $d(z)$ of the form $z_j = -a_5\zeta^j + 0(\pi^4)$ and so $r_j = z_j\pi^2$ is in L . Conversely, any root r of $\Phi(x)$ satisfies $r^5 \equiv -4a_5 \pmod{4r^2}$ and $\text{ord}_2(r) = 2/5$, so $\pi_0 = 2/r^2$ leads to a root of $a_5^2z^5 - 2$ by Hensel. By Proposition 3.5, ζ_5 is in F , so $L = F$. \square

Let $\tilde{C}: y^2 + y = \tilde{g}(x)$ be the deformation of C with \tilde{g} in equation (5.3.2) and

$$(5.3.7) \quad b_3 = a_3 + 4\epsilon_3, \quad b_j = a_j + 2\epsilon_j \text{ for } j \in \{1, 2\}, \quad b_0 = a_0 + \epsilon_0.$$

Note that the leading coefficient a_5 of \tilde{g} agrees with that of g , since it has been chosen from the discrete set of representatives for $\mathbb{W}^\times/\mathbb{W}^{\times 5}$. By the Lemma, A and \tilde{A} have the same 2-division field F . Let $\tilde{\Phi} = (4b_0 + 1)z^5 + 4b_1z^4 + 4b_2z^3 + 4b_3z^2 + 4a_5$.

Lemma 5.3.8. *For each root r of Φ , there is a root \tilde{r} of $\tilde{\Phi}$ and an ϵ_0 in \mathcal{O}_F such that $\tilde{r}/r \equiv 1 + 4\epsilon_0 \pmod{4\pi}$. If $r' \neq r$ is another root of Φ , then*

$$(\tilde{r}' - \tilde{r})/(r' - r) \equiv 1 + 4\epsilon_0 \pmod{4\pi}.$$

Let P in $A[2]$ be represented by $(r, 0) + (r', 0) - \mathfrak{O}$ and let \tilde{P} be the corresponding point in $\tilde{A}[2]$. Then $F(\frac{1}{2}\tilde{P}) = F(\frac{1}{2}P)$.

Proof. To describe the roots of $\tilde{\Phi}$ as multiples of the roots of Φ , let

$$h(z) = \frac{1}{r^5}\tilde{\Phi}(rz) = (4b_0 + 1)z^5 + \frac{4b_1}{r}z^4 + \frac{4b_2}{r^2}z^3 + \frac{4b_3}{r^3}z^2 + \frac{4a_5}{r^5}.$$

Then $h(z)$ is in $\mathcal{O}_F[z]$ and $h(z) = \frac{1}{r^5}\Phi(rz) + \delta(z)$, where

$$\delta(z) = 4\epsilon_0z^5 + \frac{8\epsilon_1}{r}z^4 + \frac{8\epsilon_2}{r^2}z^3 + \frac{16\epsilon_3}{r^3}z^2 \equiv 4\epsilon_0z^5 \pmod{4\pi\mathcal{O}_F[z]}.$$

Since $h'(1) \equiv 1 \pmod{2\pi}$ is a unit in \mathcal{O}_F and $h(1) = \delta(1) \equiv 0 \pmod{4}$, Hensel's Lemma implies that there is a root s of $h(z)$ satisfying $s \equiv 1 \pmod{4}$. By quadratic convergence of Hensel iteration we find that

$$s \equiv 1 - \frac{h(1)}{h'(1)} \equiv 1 + 4\epsilon_0 \pmod{4\pi}.$$

Finally, modulo 4π , the ratio of each Kummer generator in Lemma 5.3.4 to the corresponding Kummer generator arising from \tilde{P} is $(1 + 4\epsilon_0)^4 \equiv 1 \pmod{4\pi}$ and therefore a square in F^\times . Hence $F(\frac{1}{2}\tilde{P}) = F(\frac{1}{2}P)$. \square

We briefly indicate how the Kummer groups of curves and Honda systems are related. Let \mathcal{E} be the group scheme belonging to the finite Honda system $\mathfrak{E} = \mathfrak{E}_1$ in Notation 3.4 with $k = \mathbb{F}_2$. By Proposition 3.5(iii), each non-trivial point P_a in E corresponds to a root a of $x^{15} + 8$. Let \mathcal{V} be the group scheme belonging to the finite Honda system \mathfrak{M} of exponent 4 in §5 with Honda parameters $\mathbf{s} = [\lambda; s_1, s_2, s_3, s_5]$ as

in Definition 4.5. Thus $\lambda \equiv \pm 1 \pmod{4}$ and s_j is in \mathbb{F}_2 . The extension $L_a = F(\frac{1}{2}P_a)$ of F generated by Q in V such that $2Q = P_a$ does not depend on the choice of Q . In addition, L_a/F is an elementary 2-extension whose degree divides 16.

As in the proof of Lemma 5.2.1, we found a polynomial $f_{\mathbf{s},a}(x)$ in $F[x]$ with splitting field L_a . For each a and each of the 32 Honda parameters \mathbf{s} , we used Magma to obtain the *Kummer group* for L_a , i.e. the subgroup of $F^\times/F^{\times 2}$ whose square roots generate L_a over F .

Next, we match P_a coming from the finite Honda system to a point of order 2 on the Jacobian A of the curve $C: y^2 = x\Phi(x)$ with Φ given by (5.3.5). The field $F = \mathbb{Q}_2(\pi, \zeta)$ has an explicit construction, where ζ is a primitive fifth root of unity and π is a prime element satisfying $\pi^5 = 2$. Then there is a Frobenius τ in $\text{Gal}(F/\mathbb{Q}_2)$ fixing π and a generator σ for the inertia group in $\text{Gal}(F/\mathbb{Q}_2)$ such that $\sigma(\pi) = \zeta\pi$. Since $-\pi$ is the unique root of $x^{15} + 8$ fixed by τ and the divisor $(r_5, 0) + (0, 0) - \mathfrak{O}$ represents the unique point T in $A[2]$ fixed by τ , we find that $P_{-\pi}$ and T correspond.

By Lemma 5.3.8, it suffices to determine the Honda parameters for the Jacobians of a finite set of curves. For each of these curves, we compare the Kummer group in Lemma 5.3.4 to the Kummer groups obtained by factoring $f_{\mathbf{s},-\pi}$ and thereby determine the appropriate value of \mathbf{s} .

6. GLOBAL APPLICATIONS

Throughout this section, F is the Galois closure of a favorable quintic field of discriminant $\pm 16N$, with N prime, as in Definition 1.1.

Remark 6.1. If A is a favorable abelian surface of conductor N , then $\mathbb{Q}(A[4])$ is a favorable \mathfrak{S} -field containing $F = \mathbb{Q}(A[2])$, as in Definition 1.2. Item (i) is standard, (ii) follows from Proposition 5.2.10 and (iii) is in Proposition 2.3.4 and its proof.

Since $|\mathfrak{S}| = 120 \cdot 2^{11}$, a realistic test for the existence of a favorable \mathfrak{S} -field L is desirable. We describe a subfield K' of L whose Galois closure is L , i.e. a *stem field* for L . First, let K be the subfield of F fixed by $\overline{H} = \text{Sym}\{1, 2, 3\}$, so that K is obtained by symmetrizing $r_5 - r_4$, where r_1, \dots, r_5 are the roots of a favorable quintic polynomial with splitting field F . The transposition $s = (45)$ commutes with $\text{Gal}(F/K)$ and so induces an automorphism of K . A standard double coset computation shows that there is a unique prime \mathfrak{n}_s over N in K fixed by s . Moreover, there is a unique prime \mathfrak{p} over 2 in K .

Proposition 6.2. *Let L be a favorable \mathfrak{S} -field. Then L admits four stem fields K' such that K'/K is quadratic with ray class modulus $\mathfrak{p}^6\mathfrak{n}_s\infty$.*

Proof. Let $H = \pi^{-1}(\overline{H})$ in \mathfrak{S} . There are 31 subgroups H' of index 2 in H . The action of \mathfrak{S} on left cosets of H' is faithful for 12 of them. For those, the fixed field K' of H' is a stem field for L . Let σ be a transvection in \mathfrak{S} satisfying $\pi(\sigma) = (45)$. The conjugates of σ by representatives t in \mathfrak{S} for the double cosets $Ht\langle\sigma\rangle$ generate the inertia groups at primes over N . Intersecting these inertia groups with H and H' , respectively, we find that for four choices of K'/K , the only ramification over N occurs over \mathfrak{n}_s . Definition 1.2(ii) provides the conductor bound over 2. \square

Corollary 6.3. *Let F be the Galois closure of a favorable quintic field of discriminant $\pm 16N$ and let K be the subfield of F fixed by $\text{Sym}\{1, 2, 3\}$. If no quadratic extension K'/K of modulus $\mathfrak{p}^6\mathfrak{n}_s\infty$ exists such that the Galois closure of K'/\mathbb{Q} has*

Galois group \mathfrak{S} , then no abelian surface of conductor N with 2-division field F exists.

Remark 6.4. The non-existence results in [BK3, Table 3] follow from the Corollary. Under GRH, it was used to test all favorable quintic fields in the Bordeaux tables [BT]. For 311 primes N in $\{1277, 1597, 2557, \dots, 310547, 312413\}$, we found that no favorable abelian surface of conductor N .

Corollary 6.3 admits a stronger conclusion. Suppose that B/\mathbb{Q} is a semistable abelian surface of conductor qN , with q odd and prime to N , that $\mathbb{Q}(B[2]) = F$ and that the primes over q are unramified in the 4-division field L of B . Then L is a favorable \mathfrak{S} -field. If no favorable \mathfrak{S} -field containing F exists, then all such B are ruled out, not just those with $q = 1$.

In fact, such B exist with $q > 1$. In Table 1, $[a_0, \dots, a_6]$ denotes a polynomial $f(x) = a_0 + \dots + a_5x^5$ defining a favorable quintic field. The Igusa discriminant of a minimal model for the curve $y^2 = f(x)$ is $\Delta = q^4N$ and its Jacobian B is semistable of conductor qN . Let \mathcal{B} be the Néron model of B over the strict Henselization \mathcal{O} of \mathbb{Z}_q and let $k \simeq \overline{\mathbb{F}}_q$ be the residue field of \mathcal{O} . Let \mathcal{B}_k^0 be the connected component of the identity in the special fiber \mathcal{B}_k . Then the group of connected components $\Phi = \mathcal{B}_k/\mathcal{B}_k^0$ is cyclic of order $4 = \text{ord}_q(\Delta)$ by [L, Prop. 2(ii)]. Moreover, \mathcal{B}_k^0 is the extension of an elliptic curve by a torus of dimension 1, so $\mathcal{B}_k^0[4] \simeq (\mathbb{Z}/4\mathbb{Z})^3$. Since the kernel of reduction is divisible by 4, all these points lift to $\mathcal{B}(\mathcal{O})[4]$. Hence q is unramified in $\mathbb{Q}(B[4])$.

The examples in Table 1 occurred in an old collection of abelian surfaces created by the senior author, rather than a dedicated search. They illustrate why the criterion in Corollary 6.3 often is inadequate to decide the non-existence of a favorable abelian surface of conductor N in [BK3, Table 3]. Conversely, given a favorable \mathfrak{S} -field L , is there always a semistable abelian surface B of conductor qN with $\mathbb{Q}(B[4]) = L$ for some odd integer q ?

N	q	$f(x)$
1061	3	[1, 8, 28, 56, 64, 36]
2069	31	[25, 344, 1888, 5168, 7056, 3844]
2269	3	[1, 12, 56, 124, 124, 36]
2909	3	[1, 0, -12, -4, 32, 36]
3989	11	[-11, -112, -404, -532, 68, 484]
5381	5	[1, 8, -8, 0, 0, 4]
7013	5	[1, 0, 12, 16, 20, 4]
7877	3	[1, -8, 16, 12, -56, 36]
8581	11	[-3, 52, -324, 876, -1068, 484]

TABLE 1. B has conductor qN and $\mathbb{Q}(B[4])$ is a favorable \mathfrak{S} -field

Errata. We fix a misquote in [BK2]. The orthogonal group $O_4^+(\mathbb{F}_2) \simeq \mathcal{S}_3 \wr \mathcal{S}_2$ is not generated by transvections and does not belong in Propositions 2.4 and 2.5. In Propositions 2.8 and 2.11, V should be semistable, so that Fontaine's bound on ramification at ℓ applies.

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